PROJECTIVITY OF MODULES FOR INFINITESIMAL UNIPOTENT GROUP SCHEMES

CHRISTOPHER P. BENDEL

ABSTRACT. In this paper, it is shown that the projectivity of a rational module for an infinitesimal unipotent group scheme over an algebraically closed field of positive characteristic can be detected on a family of closed subgroups.

Let k be an algebraically closed field of characteristic $p > 0$ and G be an infinitesimal group scheme over k, that is an affine group scheme G over k whose coordinate (Hopf) algebra $k[G]$ is a finite-dimensional local k-algebra. A rational G-module is equivalent to a $k[G]$-comodule and further equivalent to a module for the finite-dimensional cocommutative Hopf algebra $k[G]^* \equiv \operatorname{Hom}_k(k[G], k)$. Since $k[G]^*$ is a Frobenius algebra (cf. [**Jan**]), a rational G-module (even infinite-dimensional) is in fact projective if and only if it is injective (cf. [**FW**]). Further, for any rational G-module M and any closed subgroup scheme $H \subset G$, if M is projective over G, then it remains projective upon restriction to H (cf. [**Jan**]). We consider the question of whether there is a "nice" collection of closed subgroups of G upon which projectivity (over G) can be detected.

For an example of what we mean by a "nice" collection, consider the situation of modules over a finite group. Over a field of characteristic $p > 0$, a module over a finite group is projective if and only if it is projective upon restriction to a p-Sylow subgroup (cf. [**Rim**]). For a p-group (and hence for any finite group), L. Chouinard [**Ch**] showed that a module is projective if and only if it is projective upon restriction to every elementary abelian subgroup. If the module is assumed to be finite-dimensional, this result follows from the theory of varieties for finite groups (cf. [**Ca**] or [**Ben**]). Indeed, elementary abelian subgroups play an essential role in this theory.

In work of A. Suslin, E. Friedlander, and the author [**SFB1**], [**SFB2**], a theory of varieties for infinitesimal group schemes was developed. In this setting, subgroups of the form $\mathbb{G}_{a(r)}$ (the rth Frobenius kernel of the additive group scheme $\mathbb{G}_a$) play the role analogous to that of elementary abelian subgroups in the case of finite groups. Not surprisingly then, for finite-dimensional modules, one obtains the following analogue of Chouinard's Theorem.

Proposition 1 ([**SFB2**, Proposition 7.6])**.** *Let k be an algebraically closed field of characteristic $p > 0$, $r > 0$ be an integer, G be an infinitesimal group scheme over k of height $\leq r$, and M be a finite-dimensional rational G-module. Then M is projective as a rational G-module if and only if whenever $H \subset G$ is a subgroup scheme isomorphic to $\mathbb{G}_{a(s)}$ (with $s \leq r$) the restriction of M to H is projective as a rational H-module.*

Under some stronger hypotheses, E. Cline, B. Parshall, and L. Scott had previously shown that it suffices to consider a much smaller collection of subgroups (cf. [**CPS**, Main Theorem]). More precisely, their result applies to a group scheme G of the form $N_{(r)}$ (the rth Frobenius kernel of N) where N is a connected (reduced) T-stable subgroup scheme of a connected, semisimple algebraic group scheme over k (and T is a maximal torus). If M is a finite-dimensional $N_{(r)}T$-module (i.e. M is an $N_{(r)}$-module which admits a compatible T-structure), then the projectivity of M may be detected by taking only those subgroup schemes of the form $H = U_{\alpha(r)}$ for each root subgroup $U_\alpha \subset N$. However, this result does not hold in general for infinite-dimensional modules (cf. [**CPS**, Example (3.2)]), whereas Chouinard's Theorem for finite groups does.

The goal of this paper is to show that Proposition 1 also holds for infinite-dimensional modules if G is assumed to be *unipotent*. (An affine group scheme G is said to be unipotent if it admits an embedding as a closed subgroup of U_n, the subgroup scheme in GL_n of strictly upper triangular matrices, for some postive integer n.) Although unnecessary for Chouinard's Theorem, to handle arbitrary modules, we need a slightly stronger hypothesis. Indeed, consider the case that G is isomorphic to a product $\mathbb{G}_{a(1)}^{\times n}$ of height one additive group schemes. As an algebra, $k[G]^*$ is isomorphic to the group algebra of an elementary abelian p-group and the desired theorem becomes equivalent to Dade's Lemma [**Dade**] which says that a $k\Pi$-module for an elementary abelian p-group Π is projective if and only if it is projective upon restriction to every "cyclic shifted subroup". This result was originally proved for finite-dimensional modules and D. Benson, J. Carlson, and J. Rickard [**BCR2**] observe that one must consider a larger field in order for Dade's Lemma to hold in general. As such, we must also consider field extensions. Specifically, in Section 2, we prove the following theorem using some of the ideas in [**SFB2**] and an argument similar to that of Chouinard [**Ch**] without appealing to the theory of varieties.

Theorem. *Let k be an algebraically closed field of characteristic $p > 0$, $r > 0$ be an integer, and G be an infinitesimal unipotent group scheme over k of height $\leq r$. For any rational G-module M, M is projective as a rational G-module if and only if for every field extension K/k and every (closed) K-subgroup scheme $H \subset G \otimes_k K$ with $H \simeq \mathbb{G}_{a(s)} \otimes_k K$ (with $s \leq r$) the restriction of $M \otimes_k K$ to H is projective as a rational H-module.*

The essential property of a unipotent group scheme is that the trivial module k is the only simple module or equivalently that $\operatorname{Hom}_G(k, M) = M^G \neq 0$ for all non-zero rational G-modules M. (Indeed, this property is sometimes taken as the definition of a unipotent group scheme.) Observe that the theory of finite-dimensional algebras shows that if k is the only simple module for such an algebra, then the algebra is indecomposable as a module over itself, and hence a module is in fact projective if and only if it is free. Hence, over an infinitesimal unipotent group scheme, the notions of projective, injective, and free are all equivalent.

In the sense of these properties, infinitesimal unipotent group schemes are analogues of finite p-groups. For finite groups, the key to reducing the projectivity of a G-module to the case of a p-group is that the restriction map in cohomology induced by the embedding of a p-Sylow subgroup is an injection. Unfortunately, there is no analogous result in the setting of group schemes.

We remind the reader that the *restricted* representation theory of a restricted Lie algebra $\mathfrak{g}$ over k is equivalent to the representation theory of a certain (height 1) infinitesimal group scheme (cf. [**Jan**]). As such, the theorem applies to *p-nilpotent* restricted Lie algebras, which may be embedded in a Lie algebra of strictly upper triangular matrices, and may be stated as follows.

Corollary. *Let k be an algebraically closed field of characteristic $p > 0$, $\mathfrak{g}$ be a finite-dimensional, p-nilpotent restricted Lie algebra over k, and M be a $u(\mathfrak{g})$-module, where $u(\mathfrak{g})$ denotes the restricted enveloping algebra of $\mathfrak{g}$. Then M is projective over $u(\mathfrak{g})$ if and only if for every field extension K/k, M is projective upon restriction to each subalgebra $u(\langle x \rangle) \subset u(\mathfrak{g} \otimes_k K)$ for all $x \in \mathfrak{g} \otimes_k K$ with $x^{[p]} = 0$, where $\langle x \rangle \subset \mathfrak{g} \otimes_k K$ denotes the one-dimensional restricted Lie subalgebra of $\mathfrak{g} \otimes_k K$ spanned by x, and $x^{[p]}$ denotes the image of x under the restriction map on $\mathfrak{g} \otimes_k K$.*

In this context, the result of Cline, Parshall, and Scott asserts that it suffices to consider only those elements x which are root vectors. Consider the two-dimensional nilpotent Lie subalgebra $\mathfrak{g} = \langle x_\alpha, x_{\alpha+\beta} \rangle \subset \mathfrak{sl}_3$ generated by root vectors, where α and β are the simple roots. In Example (3.2) of [**CPS**], they exhibit an infinite-dimensional non-projective $\mathfrak{g}$-module M which is free upon restriction to the root vectors x_α and $x_{\alpha+\beta}$; contradicting an infinite-dimensional version of their result. On the other hand, the interested reader can readily check that this module M is not free upon restriction to the element $x = cx_\alpha + dx_{\alpha+\beta}$ whenever $c, d \in k$ are both non-zero and hence does not contradict the corollary.

Before turning to the proof, we note one potential application of the theorem (or preferably a generalization of it to arbitrary infinitesimal group schemes). Recently D. Benson, J. Carlson, and J. Rickard [**BCR1**], [**BCR2**] have developed a theory of varieties for infinitely generated modules over finite groups. While Chouinard's Theorem for finite-dimensional modules follows from the theory of varieties, it turns out that knowing Chouinard's Theorem in advance for arbitrary modules is necessary for certain results in the more general theory. Hence, an attempt to generalize the theory of varieties to arbitrary modules for infinitesimal group schemes may in part require the above theorem.

1. COHOMOLOGY FACTS

In this section, we record two results about cohomology which will be used in the next section to prove the theorem. First, the following well known fact relates projectivity to vanishing of cohomology for unipotent group schemes. Indeed, the reader will readily observe that this result holds for modules over an arbitrary finite-dimensional k-algebra which admits k as the only simple module and has the property that the notions of injective and projective are equivalent.

Proposition 2. *Let k be a field of characteristic $p > 0$ and G be an infinitesimal unipotent group scheme over k. For any rational G-module M, M is projective $(= injective = free)$ if and only if $H^1(G, M) = 0$.*

To prove Proposition 2, we need to consider a *minimal* injective resolution I_* for M. Any rational G-module M has a unique up to isomorphism *injective hull* Q_M such that $\operatorname{soc} Q_M \simeq \operatorname{soc} M$ (cf. [**Jan**, I 3.17]). We can form as usual an injective resolution of M:

$$I_0 \xrightarrow{\delta_0} I_1 \xrightarrow{\delta_1} I_2 \to \cdots$$

by taking $I_0 = Q_M$ and inductively I_n to be the injective hull of $I_{n-1}/\operatorname{im} \delta_{n-2}$ and $\delta_{n-1} : I_{n-1} \twoheadrightarrow I_{n-1}/\operatorname{im} \delta_{n-2} \hookrightarrow I_n$ (with $I_{-1} \equiv M$ and $\delta_{-1} : M \hookrightarrow Q_M$). Evidently we have $I_{n-1}/\operatorname{im} \delta_{n-2} \simeq \operatorname{im} \delta_{n-1} = \ker \delta_n$.

Lemma 1 (cf. [**Ben**, I 2.5.4]). *Let I_* be the minimal injective resolution of M constructed above. Then the differential in the complex $\operatorname{Hom}_G(k, I_*)$ is trivial and hence we have $H^i(G, M) = \operatorname{Hom}_G(k, I_i) = I_i^G$ for all $i \geq 0$.*

Proof. Consider $\delta_i : I_i \to I_{i+1}$ for any $i \geq 0$ and let $\alpha : k \to I_i$ be a nonzero map. We want to show that $\delta_i \circ \alpha = 0$. Since α is a nonzero map and k is simple, α is in fact injective and so $\alpha(k)$ is a simple submodule of I_i. Hence, $\alpha(k)$ is contained in the socle of I_i. By construction, $\operatorname{soc} I_i = \operatorname{soc}(\ker \delta_i) \subset \ker \delta_i$. Hence, $\alpha(k) \subset \ker \delta_i$. In other words, $\delta_i \circ \alpha = 0$ as claimed. $\qquad\square$

Using Lemma 1, we now prove Proposition 2.

Proof of Proposition 2. If M is projective, then clearly $H^i(G, M) = 0$ for all $i > 0$. Conversely, suppose that $H^1(G, M) = 0$ and let I_* be a minimal injective resolution of M as above. Lemma 1 shows in particular that $H^1(G, M) = I_1^G$ and hence $I_1^G = 0$. Since G is unipotent, the G-fixed points of every non-zero module are non-zero. Hence, we must have $I_1 = 0$. But, that means M was injective (equivalently projective) to begin with. $\qquad\square$

In the proof of the theorem, we will also need to make use of the structure of the cohomology algebra of $\mathbb{G}_{a(1)}$, which is simply the same as the cohomology algebra of the finite cyclic group $\mathbb{Z}/p$.

Proposition 3 ([**CPSvdK**]). *If $p \neq 2$, then the cohomology algebra $H^*(\mathbb{G}_{a(1)}, k)$ is the tensor product of a polynomial algebra $k[x_1]$ in one generator x_1 of degree 2 and an exterior algebra $\Lambda(\lambda_1)$ in one generator λ_1 of degree 1. If $p = 2$, then $H^*(\mathbb{G}_{a(1)}, k) = k[\lambda_1]$ is a polynomial algebra in one generator λ_1 of degree 1 (and in this case we set $x_1 = \lambda_1^2$).*

2. Proof of Theorem

We now proceed to prove the theorem. If M is projective as a rational G-module, then $M \otimes_k K$ is projective over $G \otimes_k K$ and, as already noted, remains so upon restriction to H. Conversely, suppose all restrictions are projective. Let k'/k be any (even trivial) algebraically closed field extension, $G' = G \otimes_k k'$, and $M' = M \otimes_k k'$. Clearly for any field extension K/k' and K-subgroup scheme $H \subset G' \otimes_{k'} K$ with $H \simeq \mathbb{G}_{a(s)} \otimes_k k' \otimes_{k'} K$ the restriction of $M' \otimes_{k'} K$ to H is projective as a rational H-module.

We proceed by induction on $\dim_{k'} k'[G']$ to show the slightly stronger statment that M' is projective over G' for any such k'. If $\dim_{k'} k'[G'] = 1$, G' is the trivial group scheme and there is nothing to prove. So, assume now that the result holds for any k' and any $H \subset G'$ with $\dim_{k'} k'[H] < \dim_{k'} k'[G']$. If $G' \simeq \mathbb{G}_{a(s)} \otimes_k k'$ for some $s \leq r$, then we are done by assumption.

So, we may assume from now on that $G' \not\simeq \mathbb{G}_{a(s)} \otimes_k k'$. By Proposition 2, to show that M' is projective over G', it suffices to show that $H^1(G', M') = 0$. Let $\phi : G' \twoheadrightarrow \mathbb{G}'_{a(1)} = \mathbb{G}_{a(1)} \otimes_k k'$ be any non-trivial homomorphism of group schemes. (As G is unipotent, there exists at least one such homomorphism.) Consider the short exact sequence of group schemes over k'

$$1 \to N \to G' \to \mathbb{G}'_{a(1)} \to 1$$

and the induced Hochschild-Serre spectral sequence

$$E_2^{p,q}(M') = H^p(\mathbb{G}'_{a(1)}, H^q(N, M')) \Rightarrow H^{p+q}(G', M').$$

Now, ϕ induces a map on cohomology $\phi^* : H^*(\mathbb{G}'_{a(1)}, k') \to H^*(G', k')$. Let $x_\phi = \phi^*(x_1)$ where $x_1 \in H^2(\mathbb{G}'_{a(1)}, k')$ is the canonical generator. Furthermore, the spectral sequence admits an action of the cohomology ring $H^*(\mathbb{G}'_{a(1)}, k')$, with $H^*(\mathbb{G}'_{a(1)}, k')$ acting on the abutment via ϕ^*.

Since ϕ is non-trivial, $\dim_{k'} k'[N] < \dim_{k'} k'[G']$ and so by induction M' is projective upon restriction to N. Thus, $H^i(N, M') = 0$ for all $i > 0$ and the spectral sequence collapses to

$$E_2^{p,0} = H^p(\mathbb{G}'_{a(1)}, (M')^N) \Rightarrow H^p(G', M')$$

giving an isomorphism $H^*(\mathbb{G}'_{a(1)}, (M')^N) \simeq H^*(G', M')$. We now recall that the action of x_1 induces a periodicity isomorphism $H^i(\mathbb{G}'_{a(1)}, Q) \simeq H^{i+2}(\mathbb{G}'_{a(1)}, Q)$ for all $i > 0$ and any rational $\mathbb{G}'_{a(1)}$-module Q (cf. [**SFB2**, 2.3]). Hence, the action of $x_\phi \in H^2(G', k')$ also induces a periodicity isomorphism $H^i(G', M') \simeq H^{i+2}(G', M')$ for all $i > 0$.

We consider the following two cases.

CASE I: $\dim_{k'} \mathrm{Hom}_{Gr/k'}(G', \mathbb{G}'_{a(1)}) = 1$

By Theorem 1.6 of [**SFB2**] (which is an analogue of a characterization of p-groups in terms of cohomology by J.-P. Serre [**Se**]), $x_\phi \in H^2(G', k')$ is nilpotent and so we conclude from the above periodicity isomorphism that $H^1(G', M') = 0$ and hence M' is projective.

CASE II: $\dim_{k'} \mathrm{Hom}_{Gr/k'}(G', \mathbb{G}'_{a(1)}) > 1$

Let ϕ_1 and ϕ_2 be two linearly independent homomorphisms from G' onto $\mathbb{G}'_{a(1)}$. Further, let $c_1, c_2 \in k'$. Then, by Corollary 1.5 of [**SFB2**],

$$c_1 x_{\phi_1} + c_2 x_{\phi_2} = x_{c_1^{1/p} \phi_1} + x_{c_2^{1/p} \phi_2} = x_{c_1^{1/p} \phi_1 + c_2^{1/p} \phi_2} \in H^2(G', k').$$

If at least one of c_1, c_2 is non-zero, the map $c_1^{1/p} \phi_1 + c_2^{1/p} \phi_2 : G' \to \mathbb{G}'_{a(1)}$ is non-zero and so the action of $c_1 x_{\phi_1} + c_2 x_{\phi_2}$ induces a periodicity isomorphism $H^i(G', M') \simeq H^{i+2}(G', M')$ for all $i > 0$. In particular, $c_1 x_{\phi_1} + c_2 x_{\phi_2} : H^1(G', M') \xrightarrow{\sim} H^3(G', M')$. Hence, if $H^1(G', M')$ was a finite-dimensional space, then one would readily conclude from an eigenvalue argument that $H^1(G', M') = 0$. Since $H^1(G', M')$ might be infinite-dimensional (as M' might be), we make use of an infinite-dimensional substitute used in [**BCR2**]. For the reader's convenience, we restate this result here.

Lemma 2 ([**BCR2**, Lemma 4.1]). *Let V and W be vector spaces over an algebraically closed field k and K be a non-trivial field extension of k. Suppose that $\phi_1, \phi_2 : V \to W$ are linear maps with the property that for every pair of scalars*

$c_1, c_2 \in K$, *not both zero, the linear map* $c_1\phi_1 + c_2\phi_2 : V \otimes_k K \to W \otimes_k K$ *is an isomorphism. Then both* V *and* W *are the zero vector space.*

Continuing with the proof of the theorem, let K/k' be any non-trivial algebraically closed field extension. Consider the base changes of G' and M' to K: $G_K \equiv G' \otimes_{k'} K$ and $M_K \equiv M' \otimes_{k'} K$. Then we have $H^*(G', k') \otimes_{k'} K \simeq H^*(G_K, K)$ and $H^*(G', M') \otimes_{k'} K \simeq H^*(G_K, M_K)$. Further, since ϕ_1 and ϕ_2 remain linearly independent after base change, for any $c_1, c_2 \in K$ with at least one non-zero, the map $c_1\phi_1 + c_2\phi_2 : G_K \to \mathbb{G}'_{a(1)} \otimes_{k'} K$ is non-trivial. Since our inductive assumption applies to any algebraically closed field extension of k, it also applies to K and so as above we may conclude that

$$c_1 x_{\phi_1} + c_2 x_{\phi_2} = x_{c_1^{1/p}\phi_1 + c_2^{1/p}\phi_2} : H^1(G_K, M_K) \xrightarrow{\sim} H^3(G_K, M_K).$$

Hence, applying Lemma 2 to K/k', we again conclude that $H^1(G', M') = 0$ and so M' is projective over G' and the proof is complete.

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Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556

Current address: Department of Mathematics, Statistics, and Computer Science, University of Wisconsin-Stout, Menomonie, WI 54751

E-mail address: bendelc@uwstout.edu